Chapter 8.1 part 1

8.1 Lagrange Theorem

Group Action

G - a group $\quad S$ - a set

Def G is acting on S (left action) means that we have a homomorphism

$$G \to A(S)$$
$$g \mapsto \varphi_g$$

$A(S) = \{ f : S \to S \mid f \text{ is bijective} \}$

group operation is the group operation is the composition of functions

Group homomorphism: for every $a \in S$ for every $g_1, g_2 \in G$ $\quad (\varphi_{g_1} \circ \varphi_{g_2})(a) = \varphi_{g_1 g_2}(a)$

$\varphi_{g_1}(\varphi_{g_2}(a)) = \varphi_{g_1 g_2}(a)$

Notation: Instead of $\varphi_g(a)$ we will write $g \cdot a$ (ga sometimes)

$$g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$$

$$g_1 g_2 a = g_1 g_2 a$$

Examples

D_4 acts on the square

S_n acts on $\{1, \dots, n\}$

G acts on itself

$G \to A(G)$

$g \mapsto \varphi_g$ $\qquad \varphi_g : G \longrightarrow G$

$\qquad\qquad\qquad x \longmapsto gx$

$\varphi_g(x) = \underline{gx}$

left regular representation

<u>Terminology</u>

G is acting on S

For $x \in S$ the set $\{g \cdot x \mid g \in G\} \subseteq S$ is called <u>the orbit</u> of x

<u>Th</u> The relation on S defined by

$x \sim y$ iff x and y belong to the same orbit

is an <u>equivalence relation</u>

<u>Pf</u> - reflexive $x \sim x$ because $e \cdot x = x$

for every $x \in S$

homomorphism $G \to A(S)$ takes identity $e \in G$ to the identity in $A(S)$, which is the identity map.

- symmetric

if $x \sim y$ then $y \sim x$

If x is on the orbit of y, then y is on the orbit of x

$x = g \cdot y, \quad g \in G$

act with g^{-1}:

$g^{-1} \cdot x = g^{-1} \cdot (g \cdot y)$

$\underline{g^{-1} \cdot x} = g^{-1} g \cdot y = e \cdot y = \underline{y}$

$y = g^{-1} \cdot x$

- transitive $\left.\begin{matrix} x \sim y \\ y \sim z \end{matrix}\right\}$ imply $x \sim z$

$\left.\begin{matrix} x = g \cdot y \\ y = h \cdot z \end{matrix}\right| \; x = g \cdot (h \cdot z)$

$x = gh \cdot z \qquad gh \in G$

<u>Cor</u> If G acts on S, then S is partitioned into a union of orbits (equivalence classes)

Ex Let K be a subgroup of G

K acts on G $\qquad K \longrightarrow A(G)$ $\qquad k\cdot g = kg$

$k \longmapsto \varphi_k = (g \longmapsto kg)$

Orbits

$Ka = \{ka \mid k \in K\} \subseteq G$ orbit of $a \in G$

In particular, the orbit of identity $e \in G$ is

$\{ke \mid k \in K\} = \{k \mid k \in K\} = K$

Specific Example $\quad G = \mathbb{Z}$ has a subgroup $\{5m \mid m \in \mathbb{Z}\} = K$

Action $\quad 5m \in K \qquad \mathbb{Z} \longrightarrow \mathbb{Z}$ $\qquad$ The action becomes

$i \longmapsto 5m + i$ $\qquad$ a shift by $5m$

Orbits - the orbit of $x \in \mathbb{Z}$ is

$\{5m + x \mid m \in \mathbb{Z}\} = \{y \mid y \equiv x \pmod 5\}$

- congruence class modulo 5 of x - $[x]_5$

$\underline{Ka = \{ka \mid k \in K\}}$ - orbits (equivalence classes)

The set - the group G is partitioned into the orbits

$$G = \bigcup_{a \in G} Ka$$

This partition of a group into non-overlapping orbits is called right coset decomposition

The orbits – Ka – <u>cosets</u>

Left action of a subgroup $K \subset G$ on the group G yields right coset decomposition

K is one of the cosets (the orbit of $e \in G$)

<u>All cosets are of the same size</u>

a coset $Ka = \{ka \mid k \in K\} \subseteq G$

The map $K \longrightarrow Ka$
$k \longmapsto ka$ is a bijection of sets: multiplication from the right by a^{-1} performs the inverse map

$kaa^{-1} = k$

$K \longleftarrow Ka$

$k \longleftarrow\!\shortmid ka$

If K is finite, then, for every $a \in G$, <u>Ka</u> has the same amount of elements as <u>K</u>

Def Index $[G:K]$ of a subgroup (K) in a group (G)

- the number of different cosets in the decomposition

$$G = \bigcup_{a \in G} Ka$$

Lagrange thm

Th 8.5 Assume that G is a finite group.

$$|G| = |K| \, [G:K]$$

$|G|$ is the order of G